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Scaling Assumption for Lattice Animals in Percolation Theory

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A scaling assumption for the number g_{ns} of different cluster configurations with perimeter s and size n leads to the desired cluster numbers near the percolation threshold. The perimeter distribution function has a mean square width proportional to n for large n. The relation between the average perimeter and the cluster size n for percolation has three different forms at p_c, below p_c, and above p_c and is closely related to the shape of the cluster size distribution.

KEY WORDS: Percolation; phase transitions; animals; clusters; perimeter.

1. INTRODUCTION

In the percolation problem, each site of an infinite lattice is randomly occupied or empty with probabilities p and $1 - p$, respectively. The occupied sites can be grouped into finite clusters, defined as sets of occupied sites connected by nearest neighbor distances. For p above some percolation threshold p_c, an infinite cluster exists besides the many finite clusters. To describe microscopically the phase transition and the critical exponents near p_c one needs to know the average number $c_n = c_n(p)$ of clusters containing n occupied sites each. This number c_n again can be calculated exactly if one knows the number g_{ns} of geometrically different cluster configurations or "animals"(1) corresponding to the same size n and perimeter s. This perimeter s is the number of empty lattice sites that are nearest neighbors to occupied cluster sites. In this sense the pure geometrical problem of determining the animal numbers g_{ns} is more fundamental than the question of p-dependent cluster numbers c_n. The present paper suggests an approximation for the number g_{ns} of animals; this assumption then leads to a consistent description of the percolation phase transition via the cluster numbers c_n.

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Assumptions on the number g_{ns} of animals were made earlier by Leath⁽²⁾ and Domb.⁽¹⁾ The problems connected with these assumptions are discussed in Refs. 1 and 3; in particular they did not lead to a general scaling form for the cluster numbers c_n. (See also Refs. 4–8.) Therefore the present work further generalizes these earlier attempts and succeeds in deriving the general scaling form for the cluster numbers, as desired, from a similar scaling form for the animal numbers g_{ns}.

Section 2 reviews some properties of the cluster size distribution c_n for percolation and gives a heuristic "derivation" of our basic assumption on g_{ns}. In Section 3 we derive from the g_{ns} the cluster numbers c_n; thus this section is the most important part. Section 4 discusses the resulting form of the average perimeter of percolation clusters as a function of cluster size n and concentration p. We also give a somewhat different derivation of the result of Section 3 for the mean square width of the perimeter distribution function. In concluding, Section 5 suggests possible tests of the present suggestions.

2. THE BASIC ASSUMPTION

The known scaling assumption for the cluster numbers c_n is reviewed in this section, and a new scaling assumption for the animal numbers g_{ns} is made plausible.

Let c_{ns} be the average number (per lattice site) of percolation clusters with size n and perimeter s; then $c_n = \sum_s c_{ns}$ is the total number of clusters with size n. Similarly, g_{ns} is the number of geometrically different cluster configurations (per lattice site) with size n and perimeter s; and $g_n = \sum_s g_{ns}$ is the total number of animals with fixed size n. The perimeter distribution functions thus are c_{ns}/c_n for percolation clusters and g_{ns}/g_n for animals. Clusters and animals are connected through the exact result⁽⁹⁾

$$c_{ns} = g_{ns}(1-p)^s p^n \tag{1}$$

If S and E denote entropy and energy in statistical physics, then in some sense g_{ns} corresponds to e^{S/k_B} and $p^n(1-p)^s$ to e^{-E/k_BT}; for then we have $c_{ns} = e^{-F/k_BT}$ with a free energy $F = E - TS$. But this analogy is not exact, since the perimeter of percolation clusters is not identical to the energy of a cluster in the Ising magnet.

Now we restrict ourselves to the phase transition region $n \to \infty$, $p \to p_c$, where a scaling form for the percolation cluster numbers c_n has been suggested⁽⁴⁾:

$$c_n \propto n^{-\tau} f(\epsilon n^{\sigma}) \tag{2}$$

where $\epsilon \propto p - p_c$, $\tau = 2 + 1/\delta$, and $\sigma = 1/\beta\delta$ in the usual notation of critical exponents; for example, $\tau \simeq 2.054$ and $\sigma \simeq 0.39$ in two dimensions.⁽⁶⁾

The scaling function $f = f(x)$ in (2) is analytic in x and finite at $x = 0$ [thus $c_n(p_c) \propto n^{-\tau}$]; $f(x)$ has a maximum at some negative argument(3,4) corresponding to p below p_c, and $f(x)$ decays rapidly away from this maximum. More precisely, an exponent ζ can be defined through

$$\log c_n \propto -n^{\zeta} \qquad (n \to \infty, \quad p \neq p_c) \tag{3a}$$

which means

$$\log f \propto -|x|^{\beta\delta\zeta} \qquad (x = \epsilon n^{\sigma} \to \pm\infty) \tag{3b}$$

Then we have in d dimensions presumably(3,5,7,8)

$$\zeta(p < p_c) = 1 \tag{3c}$$

$$\zeta(p > p_c) = 1 - 1/d \tag{3d}$$

Below p_c each occupied site belongs to some finite cluster: $\sum_n c_n n = p$; this condition leads to a "sum rule" for the function f,(3,4)

$$\int_0^{\infty} x^{-1-\beta}[f(x) - f(0)]\, dx = 0$$

or

$$\int_0^{\infty} x^{-\beta} f'(x)\, dx = 0 \tag{3e}$$

For animals, the g_{ns} up to $n = 20$ were tabulated by Sykes and Glen(6) for some lattices. The total number g_n of animals of size n was found to follow

$$g_n \propto n^{-\theta}\lambda^n \tag{4}$$

where, in the triangular lattice, $\theta \simeq 1$ and $\lambda \simeq 5.2$.

The average percolation perimeter $\langle s_n \rangle$ is defined as

$$\langle s_n \rangle = \sum_s s c_{ns}/c_n \tag{5a}$$

Presumably we have, at least for $p = p_c$, the relation(10,2,1)

$$\langle s_n \rangle / n = (1 - p)/p + \cdots \tag{5b}$$

for large n. With the abbreviations

$$a = s/n \tag{5c}$$

$$a_c = (1 - p_c)/p_c \tag{5d}$$

Eq. (5b) now reads

$$\langle a \rangle \to a_c \qquad \text{for} \quad p \to p_c \text{ and } n \to \infty \tag{5e}$$

This result completes our summary of known cluster properties.

As Eq. (1) indicates, the cluster numbers in percolation theory are known for all concentrations p if the p-independent numbers g_{ns} of animals are known. Our question now is: Which assumption for the animal numbers g_{ns} gives the above scaling properties and in particular Eq. (2) for the percolation problem?

Equations (2) and (4) suggest an Ansatz like

$$g_{ns} \propto n^{-y}\Lambda^n f[(a_c - a)n^\sigma] \qquad (6)$$

with some scaling function $f = f(x)$. Here $\Lambda = \Lambda(n, s)$ is determined such[2] that in $c_{ns} \propto n^{-y}\Lambda^n p^n(1 - p)^s f$ the factors $p^n(1 - p)^s$ are cancelled by Λ^n for all n and s if p is related to s and n via Eq. (5b), i.e., $(1 - p)/p = s/n = a$. In this case we have $c_{ns} \propto n^{-y}f$. This requirement for Λ is fulfilled if[1,2]

$$\Lambda = \Lambda(s/n) = (a + 1)^{a+1}/a^a \qquad (7)$$

Since Λ depends on a, the function f is not proportional to the perimeter distribution function g_{ns}/g_n (or c_{ns}/c_n). These distribution functions for s will turn out later to be Gaussians with a width $\propto\sqrt{n}$; and thus in the sums $c_n = \sum_s c_{ns}$ and $g_n = \sum_s g_{ns}$ an additional factor $n^{1/2}$ appears. Therefore the free exponent y in (6) equals $\tau + \frac{1}{2}$. We thus arrive at the following Ansatz as a generalization of the earlier attempts in Refs. 1 and 2:

$$g_{ns} \propto n^{-\tau-\frac{1}{2}}\Lambda^n f[(a_c - a)n^\sigma] \qquad (8)$$

with a, a_c, and Λ determined by Eqs. (5c), (5d), and (7).

This is our basic assumption; in the next section we show that the scaling function f appearing in (8) also appears in the number of clusters, Eq. (2). We assume (8) to be valid for those values of n and s relevant for the percolation phase transition, i.e., for large n and for a near a_c.

3. NUMBERS OF CLUSTERS AND ANIMALS

This section discusses the numbers g_n and c_n of animals and percolation clusters with a fixed size n.

From (8) we get for the total number of percolation clusters with size n

$$\begin{aligned} c_n &= \sum_s g_{ns}p^n(1 - p)^s = \int_0^\infty g_{ns}[p(1 - p)^a]^n n\, da \\ &\propto n^{-\tau-\frac{1}{2}} \int \exp\{n[(a + 1)\ln(a + 1) - a\ln a + \ln p \\ &\quad + a\ln(1 - p)]\} f[(a_c - a)n^\sigma]\, da \\ &= n^{-\tau+\frac{1}{2}} \int \exp\left[-\frac{n}{2}\left(a - \frac{1 - p}{p}\right)^2 \frac{p^2}{1 - p}\right] f[(a_c - a)n^\sigma]\, da \\ &= n^{-\tau+\frac{1}{2}} \int \exp\left[-n^2\left(a - \frac{1 - p}{p}\right)^2 \frac{1}{2\Delta_n{}^2}\right] f\, da \end{aligned} \qquad (9)$$

where higher terms of order $[a - (1 - p)/p]^3$ were neglected. Thus the exponential function in (9) has a maximum at $a = (1 - p)/p$ with a width

$$\Delta_n = [n(1 - p)/p^2]^{1/2} \tag{10}$$

in $s = an$, i.e., a width $\propto 1/\sqrt{n}$ in a, as given by Leath.⁽²⁾ On the other hand, the function f in (9) peaks at $a \simeq a_c$ with a width $\propto n^{-\sigma}$. The value of the integral (9) is therefore determined for large n by the function that has the smaller width.

We now assume

$$\sigma < \tfrac{1}{2} \tag{11}$$

an assumption valid in two dimensions ($\sigma \simeq 0.39$), presumably valid in three dimensions ($\sigma \simeq 0.48$), and also slightly below six dimensions, where⁽¹¹⁾ $\sigma = \frac{1}{2} - (1/98)(6 - d)^2 + \cdots$. Thus we exclude the "classical" exponents $\beta = \gamma = 1$ and $\sigma = \frac{1}{2}$ from our considerations and restrict ourselves to dimensionalities below six.

With that assumption, for large n the peak in the scaling function f in (9) is broader than the peak in the exponential function in (9); thus the variation of f can be neglected in the evaluation of the integral: For large n the main contribution to the integral (9) comes from $\langle a \rangle \simeq (1 - p)/p$, i.e.,

$$\langle s_n \rangle = n(1 - p)/p + \cdots \tag{12}$$

the width of the peak is $[\langle s_n^2 \rangle - \langle s_n \rangle^2]^{1/2} = \Delta_n$, as given in (10), and varies as $\sqrt{n}$. Thus Eq. (9) gives

$$c_n \propto n^{-\tau} f[(a_c - \langle a \rangle) n^\sigma] \propto n^{-\tau} f(\epsilon n^\sigma); \qquad \epsilon = (p - p_c)/pp_c \tag{13}$$

Thus Eq. (2) for the cluster numbers c_n has been derived, as promised, from Eq. (8) for the animal numbers g_{ns}; and the function $f = f(x)$ appearing in the c_n is the same function as appears in the g_{ns}.

Similarly, we now evaluate the total number g_n of animals with size n:

$$\begin{aligned} g_n = \sum_s g_{ns} &= n \int g_{ns}\, da \\ &\propto n^{-\tau+\frac{1}{2}} \int \exp\{n[(a + 1)\ln(a + 1) - a \ln a]\} f[(a_c - a)n^\sigma]\, da \\ &= n^{-\tau+\frac{1}{2}} \int \exp\{n[(a + 1)\ln(a + 1) - a \ln a + (1/n)\ln f]\}\, da \end{aligned} \tag{14a}$$

In this equation, in contrast to Eq. (9), the asymptotic behavior of the function f in Eq. (3b) also becomes important. For without the function $f = f(x)$, the exponential term would always increase with increasing a, making the integral divergent. The same effect would appear if we would use Leath's Ansatz $\ln f \simeq -n^{2\sigma}(a - a_c - n^{-\psi})^2$ (apart from constants of order unity). Equation (3c) shows that for a above a_c, i.e., for negative x, the function $f = f(x)$ decays asymptotically as $\ln f(x \to -\infty) = -c(-x)^{\beta\delta} = -c(a - a_c)^{\beta\delta} n$; thus $(1/n) \ln f$ approaches a finite value for large n. [It is crucial for this discussion that the exponent ζ in (3c) is exactly equal to unity; trivial rigorous arguments merely give $\zeta \leqslant 1$. See also Ref. 7.] Now Eq. (14a) reads

$$g_n \propto \int n^{-\tau+\frac{1}{2}} \exp\{n[(a+1)\ln(a+1) - a\ln a - c(a-a_c)^{\beta\delta}]\}\, da$$

$$= \int n^{-\tau+\frac{1}{2}}\, e^{nh(a)}\, da \tag{14b}$$

If c is a positive constant, as it is in the critical region, the function $h(a) = (a+1)\ln(a+1) - a\ln a - c(a-a_c)^{\beta\delta}$ has a single maximum at some $a = a_0 > a_c$ defined by

$$\ln\left(1 + \frac{1}{a_0}\right) = c\beta\delta(a_0 - a_c)^{\beta\delta - 1} \tag{15}$$

Again expanding quadratically around this maximum, $h(a) = h(a_0) - \text{const} \times (a - a_0)^2 + \cdots$ with some positive constant, we get

$$g_n \propto n^{-\tau} e^{-h(a_0)n} = n^{-\tau}\lambda^n \tag{16a}$$

$$\lambda = (a_0 + 1)^{a_0+1} a_0^{-a_0} \exp[-c(a_0 - a_c)^{\beta\delta}] \tag{16b}$$

in full agreement with Eq. (4). Actually, since all terms in $(1/n)\ln f$ vanishing for large n were neglected in the above derivation, Eq. (16a) is not entirely correct. The leading n dependence of g_n is still described by λ^n with λ from (16b); but higher order terms generalize (16a) to

$$g_n = \Phi_n \lambda^n; \qquad (\Phi_n)^{1/n} \to 1 \quad \text{for} \quad n \to \infty \tag{16c}$$

Equation (16c) agrees with general results$^{(1)}$ and also with the numerical fact that the exponents θ and τ in Eqs. (4) and (16a) are different.

Thus not only the evaluation of the percolation cluster numbers c_n but also the animal numbers g_n gave reasonable results: The distribution function g_{ns}/g_n for animal perimeters has a peak of width $\propto \sqrt{n}$ at some $s_0 = a_0 n$ which is larger than the average $a_c n$ for percolation perimeters at p_c. [Presumably our fundamental assumption (8) is no longer valid for the region $n \to \infty$, $a \to a_0 > a_c$ required to evaluate g_n. In that case the "constant" c of Eq. (14b) will depend on a and give corrections to Eq. (15) due to dc/da. For the cluster numbers c_n decay as $\log c_n \propto -n^{\zeta}$ not only for p near p_c, but apparently everywhere.$^{(3,7,8)}$ Analogously, we expect the term $\ln f$ in Eq. (14a) still to vary as n, even if a is not close to a_c. Since these problems would appear relevant only in a numerical evaluation of a_0 and λ, which is not undertaken here, we neglect them.]

4. MEAN VALUE AND WIDTH OF PERIMETER DISTRIBUTION FUNCTION

We derive here the next-order term in the perimeter vs. size relation of percolation clusters and discuss in greater detail the width of the perimeter distribution function.

Equations (5b) and (12) give $\langle s_n \rangle = n(1-p)/p + \cdots$, and now we want to evaluate the next term in this relation. More precisely, for fixed p near p_c we calculate the exponent of n in the leading correction term. For this purpose we can no longer neglect in the integral (9) the variation of the function f with a, as was done in deriving (12). Taking the influence of the function f as a small correction, we now have to evaluate

$$c_n \propto n^{-\tau+\frac{1}{2}} \int e^{k(a)n}\, da \tag{15a}$$

where

$$\begin{aligned} k(a) = {} & (a+1)\ln(a+1) - a\ln a + \ln p + a\ln(1-p) \\ & + (1/n)\ln f[(a_c - a)n^\sigma] \end{aligned} \tag{15b}$$

Minimization of $k(a)$ gives the condition for the mean value $\langle a \rangle$:

$$0 = k'(\langle a \rangle) = \ln(1 + 1/\langle a \rangle) + \ln(1-p) + n^{\sigma-1}\phi[(a_c - \langle a \rangle)n^\sigma]$$

with the function $\phi(x) = -d(\ln f)/dx = -f'(x)/f(x)$. For large n we find from this condition

$$\langle a \rangle = \frac{1-p}{p}\left(1 + \frac{\phi}{p}n^{\sigma-1} + \cdots\right)$$

Since

$$(a_c - \langle a \rangle)n^\sigma \simeq \left(a_c - \frac{1-p}{p}\right)n^\sigma = \epsilon n^\sigma$$

in the function ϕ apart from higher order terms, we thus find

$$\langle s_n \rangle = \frac{1-p}{p}n + \frac{1-p}{p^2}\phi(\epsilon n^\sigma)n^\sigma + \cdots \tag{16}$$

Such a structure $\langle s_n \rangle = \bar{a}n + An^\sigma$ was proposed earlier,[10] but only with a constant coefficient A.

We see now that the coefficient of the n^σ term in the perimeter depends through $\phi = -f'/f$ on the scaling variable $\epsilon n^\sigma \propto (p - p_c)n^\sigma$. Therefore, for

$n \to \infty$ and $p \to p_c$, the form of this "excess perimeter" (i.e., of the correction to the leading term $\propto n$) depends on the order in which the two limits are taken, and we have to distinguish three cases: $p = p_c$, $p > p_c$, and $p < p_c$.

If $p = p_c$ and $n \to \infty$, then $\langle s_n \rangle = n(1 - p_c)/p_c + [(1 - p_c)/p_c^2]\phi(\epsilon n^\sigma \to 0)n^\sigma$. Since the scaling function f was assumed to be analytic, its derivative $f'(0)$ exists; presumably,(3,4) it is negative; thus $\phi(0) \propto -f'(0)/f(0)$ is positive:

$$\langle s_n(p = p_c) \rangle = n(1 - p_c)/p_c + \text{const} \times n^\sigma + \cdots \tag{17a}$$

in agreement with the suggestion of Ref. 10.

If p is fixed to a value close to but not equal to p_c, and $n \to \infty$, then (16) gives $\langle s_n \rangle = n(1 - p)/p + [(1 - p)/p^2]\phi(\epsilon n^\sigma \to \pm\infty)n^\sigma$. Since in this limit $\phi = -(d/dx)\ln f \propto -(d/dx)|x|^{\beta\delta\zeta} \propto |x|^{\beta\delta\zeta - 1} = |\epsilon|^{\beta\delta\zeta - 1} n^{\zeta - \sigma}$, the "excess perimeter" varies as $n^\zeta |\epsilon|^{\beta\delta\zeta - 1}$ for sufficiently large cluster sizes n. This means that the excess perimeter $\langle s_n \rangle - n(1 - p)/p$ has for large n the same n dependence as $\log c_n$, for p both above and below p_c (but not at p_c). This conclusion agrees with assumptions in some droplet models, where the cluster numbers are proportional to exp(−const × surface area), if the surface area is identified with the excess perimeter.(10,3) Such a simple relation seems to be correct here if one looks at sufficiently large clusters, i.e., at clusters with radii larger than the coherence length ($\epsilon n^\sigma \to \pm\infty$). We also see from $\phi = -f'/f$ that $\phi(\infty)$ is positive and $\phi(-\infty)$ is negative. Thus, with the ζ from Eqs. (3c) and (3d) we find

$$\langle s_n(p > p_c) \rangle = n(1 - p)/p + \text{const} \times \epsilon^{\beta\delta(1 - 1/d) - 1} n^{1 - 1/d} \tag{17b}$$

$$\langle s_n(p < p_c) \rangle = n(1 - p)/p - \text{const}' \times (-\epsilon)^{\beta\delta - 1} n \tag{17c}$$

Therefore the excess perimeter $\langle s_n \rangle - [(1 - p)/p]n$ varies as $n^{1 - 1/d}$ above p_c, as n^σ at p_c, and as n below p_c in the present scaling theory.

In the old theory,(10) the excess perimeter always varied as n^σ and not as $n^{1 - 1/d}$; thus Domb(1) criticized correctly the identification of excess perimeter and surface area.(10) But now that identification is allowed again above p_c since Eq. (17b) gives to the excess perimeter the same n dependence as expected for the surface of a spherical droplet. This analogy to raindrops below T_c is shown schematically in Fig. 1, which suggests that the density profile of large percolation clusters ($\epsilon n^\sigma \to +\infty$) is similar to the density profile of raindrops. (This analogy(3) was recently questioned.(12)) Below p_c such a droplet model breaks down, just as for supercritical temperatures no liquid droplets surrounded by a gas phase can exist.

Another measure of the cluster "surface" is the external perimeter,(3) which does not include the perimeter due to internal holes. Leath(2) found this external average perimeter to vary roughly as n below p_c. Table I summarizes our present knowledge about these various "surfaces"; we see a

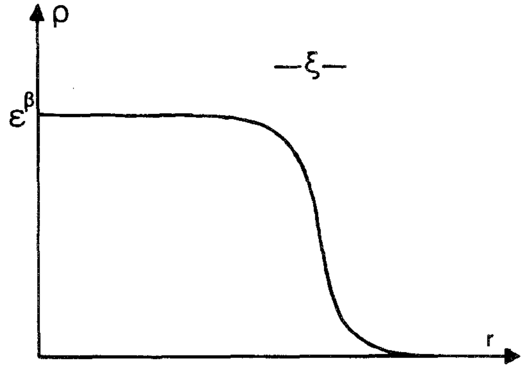

Fig. 1. Droplet picture for very large percolation clusters above the threshold p_c. The expected density profile $\rho(r)$ is plotted against the distance r from the center of mass of a large cluster, where the density is the probability that the lattice site at the place **r** belongs to the cluster. The thickness of the transition region near the droplet surface is of the order of the correlation length ξ. The value $\rho(r \to 0)$ equals the probability that an arbitrary lattice site belongs to the infinite network. Thus this picture is expected to be valid for p slightly above p_c only and for cluster radii much larger than ξ. It is suggested by the present results but not yet tested directly.

strong similarity between various quantities, suggesting for cluster radii much larger than the correlation length ξ

$$\log c_n \propto \text{surface area} \propto \text{excess perimeter} \propto \text{external perimeter}(?) \qquad (18)$$

(The external perimeter might nearly coincide with the total perimeter and thus not measure the surface in more than two dimensions if an infinite network of connected holes is percolating through all large clusters.)

Table I. Dependence on n for Various Cluster Properties if $n \to \infty$ at Fixed p[a]

	$p > p_c$	$p = p_c$	$p < p_c$
$-\log c_n$	$n^{1-1/d}$	$\log n$	n
Excess perimeter	$n^{1-1/d}$	n^σ	$-n$
External perimeter	—	—	n
Surface area	$n^{1-1/d}$	—	—
Total perimeter	n	n	n

[a] Here c_n is the number of clusters with n occupied sites. The excess perimeter is $\langle s_n \rangle - n(1-p)/p$; the external perimeter does not count internal holes. The surface area arises from the naive droplet picture of Fig. 1.

So far we have discussed the mean values of the perimeter distribution function for percolation clusters (c_{ns}/c_n) and animals (g_{ns}/g_n). These distribution functions turned out to be Gaussians centered around $a_c n$ and $a_0 n$ for large n near p_c. If we now look at the width of these distribution functions about their mean value, we see that for both percolation clusters [Eq. (10)] and for animals the width varies as $n^{1/2}$. This behavior agrees with Leath's result for large n (Monte Carlo). But from the exact perimeter polynomials for n up to 20, a width $\propto n^{0.8}$ was found in two$^{(3)}$ dimensions (also for the animals) and in three dimensions.$^{(8)}$ The present result $\Delta_n \propto n^{1/2}$ seems more reliable since it also follows from a rather different calculation.

We differentiate twice with respect to p the relation $\sum_n c_n n = p$ for $p \leqslant p_c$, mentioned before Eq. (3e). The first derivative gives$^{(2)}$

$$1 = \frac{dp}{dp} = \frac{d}{dp}\sum_n c_n n = \sum_n \sum_s g_{ns} n \frac{d}{dp}\,[p^n(1-p)^s] = \sum_n c_n n\left(\frac{n}{p} - \frac{\langle s_n\rangle}{1-p}\right)$$

Now we assume $\langle s_n\rangle = \bar{a}n + bn^z + \cdots$, with $z < 1$; here $\bar{a}$ can depend on p but is not allowed to be a scaling function of ϵn^σ, whereas b is allowed to be such a function. This assumption agrees with Eq. (16). Thus

$$1 = \left(\frac{1}{p} - \frac{\bar{a}}{1-p}\right)\sum_n n^2 c_n - \sum_n bn^{1+z}\frac{c_n}{1-p}$$

$$1 = \left(\frac{1}{p} - \frac{\bar{a}}{1-p}\right)\text{const} \times (-\epsilon)^{-\gamma} - \sum_n bn^{1+z}\frac{c_n}{1-p} \tag{19}$$

We see that $\bar{a}$ must equal $(1-p)/p$, in agreement with (12), as derived in Ref. 2. But if the coefficient $\bar{a}$ were also a scaling function of ϵn^σ, then this simple relation would break down. At present we see no reason to allow such a complication. [The sum in (19) remains finite if we take $z = \sigma$ and $b(\epsilon n^\sigma) \propto f'/f$, because of the requirement (3e) for the scaling function f. Again our approach seems to be consistent.]

With this result for $\bar{a}$, the second derivative gives, together with the definition $\Delta_n^2 = \langle s_n^2\rangle - \langle s_n\rangle^2$,

$$\begin{aligned} 0 &= \sum_n c_n n[p^{-2}(n^2 - n) + (1-p)^{-2}(\langle s_n^2\rangle - \langle s_n\rangle) - 2p^{-1}(1-p)^{-1}n\langle s_n\rangle] \\ &= \sum_n c_n n\left\{p^{-2}\left(n - \frac{p}{1-p}\langle s_n\rangle\right)^2 + (1-p)^{-2}\left[\Delta_n^2 - n\left(\frac{1-p}{p}\right)^2 - \langle s_n\rangle\right]\right\} \\ &\propto \sum_n c_n n\left[(-bn^z)^2 + \Delta_n^2 - \frac{n(1-p)}{p^2} + O(n^z)\right] \end{aligned} \tag{20}$$

Thus apparently the width term Δ_n^2 has to cancel other terms in order that the whole sum vanishes. First, the exponent z cannot be larger than $\frac{1}{2}$. For then the $(-bn^z)^2$ term would be larger than the term $\propto n$ and also could not be cancelled by the width term (since both are positive). If z is smaller than $\frac{1}{2}$, then the first term is negligible compared to the n term; and the latter has to be cancelled by the width term: Δ_n^2 must be of order n and cannot vary as $n^{1.6}$, for example. It could be a scaling function of ϵn^σ, i.e., $\Delta_n \propto n^{1/2}D(\epsilon n^\sigma)$. If the width is not such a complicated scaling function, then we must have $\Delta_n = [n(1-p)/p^2]^{1/2}$, as in Eq. (10), to fit into Eq. (20). Again our approach seems consistent. [If $z = \frac{1}{2}$, then Δ_n^2 must be of order n or smaller since now the $(-bn^z)^2$ term could be cancelled by the n term. In that case, however, it would be difficult to prevent the remaining sum in Eq. (19) from diverging, which it is not allowed to do.]

If we take $z = \sigma < \frac{1}{2}$, in agreement with our previous results, then the $(-bn^\sigma)^2$ would, for intermediate n, give important *negative* corrections to the asymptotic scaling result $\Delta_n \propto n^{1/2}$. Indeed, for $n = 14$ in the triangular lattice, our prediction $\Delta_n^2 = n(1-p)/p^2$ gives $\Delta_n^2 = 28$ at $p = p_c = \frac{1}{2}$, whereas the actual result$^{(3)}$ is only 6.01. Therefore the extrapolation from $n = 14$ to $n = \infty$ in Ref. 3 was too inaccurate for the width Δ_n.

5. CONCLUSION

This paper proposed a not too complicated expression, Eq. (8), for the number g_{ns} of geometrically different cluster configurations ("animals"). This assumption led to the desired scaling form, Eq. (13), for the number c_n of percolation clusters and also to a reasonable result, Eq. (16), for the number g_n of animals with fixed size n. A comparison with existing data on g_{ns} and c_{ns} could be made$^{(6)}$ for intermediate n. Also, Monte Carlo tests are possible: Leath$^{(2)}$ determined indirectly the g_{ns} from Monte-Carlo generated cluster numbers c_{ns}. {Apart from prefactors, his function $m(n, b/n)$ corresponds to our scaling function $f = f[(a_c - a)n^\sigma]$.} More directly, one can perhaps also work, similar to Ref. 13, with clusters of a fixed size n and change only randomly the cluster shape. If the statistical weight of different cluster shapes varies as $(1-p)^s$, then one finds in this Monte Carlo simulation the percolation perimeter distribution c_{ns}/c_n; if instead all configurations are given the same weight, one finds the animal perimeter distribution g_{ns}/g_n. In such a calculation one could better check some of our assertions here, e.g., that the excess perimeter $\langle s_n \rangle - n(1-p)/p$ varies for large n as n, n^σ, and $n^{1-1/d}$ below, at, and above p_c [Eq. (17)]. Or does the average animal perimeter indeed approach a value $a_0 n$ which is larger than the average percolation cluster perimeter at p_c [Eq. (15)]? Moreover, our table has "holes" to be filled in for the external perimeter.

More generally, it would be important to find the density profile of large clusters and to see whether above p_c it agrees with the schematic picture of our figure if one averages over enough clusters of the same size. A study of random walks within a cluster (ant-in-a-labyrinth problem) could give a first answer.(14) The same density profile could also be studied for animals; no prediction was dared here.

Analytically, one could search for alternatives to the present suggestions, that give the same desired results but on a different foundation. Even if the present form is accepted as valid asymptotically for large clusters, one needs corrections to describe more accurately the behavior at intermediate cluster sizes, for which perimeter polynomials exist.(6)

NOTE ADDED IN PROOF

Preprints of M. Schwartz, G. R. Reich, and P. L. Leath and of J. Hoshen and D. Stauffer confirm some of the present results by simpler, more general, or more reliable derivations. For example, $\langle s_n \rangle = n(1-p)/p - (1-p)d(\ln c_n)/dp$ and a similar formula for Δ_n hold exactly for all p and thus generalize our results of Section 4.

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